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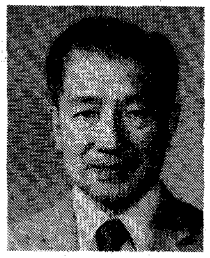
Chang-Hong Liang was born in 1943 in Shanghai, China. He graduated from the Department of Electromagnetic Engineering, Northwest Telecommunication Engineering Institute, Xian, People's Republic of China, in July 1965. After graduation he remained at the Institute as a Graduate Assistant and then as a Lecturer. He has done research work in the areas of ferrite digital phase shifter, phased antenna arrays, and microwave directional couplers. In August 1980 he came to the United States as a Visiting Scholar, under the guidance of Professor D. K. Cheng of Syracuse University.



David K. Cheng (S'44–A'48–SM'50–F'60) obtained his B.S. degree in electrical engineering from the National Chiao-Tung University in China. He went to the United States in 1943 for postgraduate studies at Harvard University where he received the S.M and Sc.D. degrees in 1944 and 1946, respectively. While at Harvard, he was a Charles Storrow Scholar and a Gordon McKay Scholar.

Dr. Cheng was associated with the Communications Laboratory of the USAF Cambridge Research Laboratories from 1946 to 1948. In 1948 he joined the faculty of Syracuse University where he is a Professor of Electrical Engineering. He was honored by Syracuse University as a Centennial Professor in 1970, and received the Chancellor's Citation in 1981 for exceptional academic achievement.

Dr. Cheng has published one book, *Analysis of Linear Systems* (Addison-Wesley), in 1959, and numerous journal articles on electromagnetic theory, antennas and arrays, and communication and signal-processing systems. Another book, *Field and Wave Electromagnetics*, is due to be published in 1982. He was a member of the IEEE Publications Board for 1968–1970; a member of the Administrative Committee of IEEE Antennas and Propagation Society, 1972–1974; the Consulting Editor of an Electrical Engineering Monograph series for the Intext Educational Publishers, 1969–1972; and a Consulting Editor for Electrical Science books published by the Addison-Wesley Publishing Co., 1961–1978.

In 1960–1961 Dr. Cheng was awarded a Guggenheim Fellowship to study and travel in Europe. For the 1975–1976 year he served as a Liaison Scientist of the London Branch of the Office of Naval Research, on leave from Syracuse University. The IEEE Antennas and Propagation Society appointed him as the European Lecturer under its Distinguished Lecturer Program for the same period.

An Alternative Theory of Optical Waveguides with Radial Inhomogeneities

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Abstract—**The field equations are solved for an inhomogeneous dielectric cylinder with azimuthal symmetry. The solutions are shown to satisfy particular orthogonality relations and allow derivation of simple, generally valid expressions for dispersion relation, power flow, energy density, and group delay. A method for numerical solution of the equations, the modified staircase method, is proposed. It is shown that it leads to expressions similar to those of the Wentzel-Kramer-Brillouin (WKB) method, but, unlike the latter, is valid for the lowest order guided modes. The method has been tested in a computer program.**

Manuscript received October 2, 1981; revised November 11, 1981. This work was supported by the Royal Norwegian Council for Scientific and Industrial Research.

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I. Introduction

THE PRESENT paper presents a theory of wave propagation on an inhomogeneous optical waveguide of cylindrical symmetry. The discussion is based on a formulation of the field equations as a single, first-order differential equation in a four-dimensional vector space. Similar formulations have been used by several authors as a basis for numerical field calculations [1], [2]. Vigants and Schlesinger [3] in a pioneering paper argue for this type of approach and point out the advantages obtained by treating the field equations as a set of first-order equations when use is made of existing mathematical techniques for handling sets of equations. Confining their discussion to higher order modes, these authors derive the characteristic equation for the wave propagation in a radially inhomogeneous cylinder, and solve numerically the HE_{21} and EH_{21} mode propagation for particular models of index variation. In the present paper, emphasis is on deriving a number of analytical results. It is shown that this may be done without introduction of approximations, provided that the vector space is so chosen that the inherent symmetries of Maxwell's equations are preserved. It is believed that the method is a useful alternative to the more commonly used procedure of transforming the field equations into two coupled second-order equations. Examples of the applications of the analytical results in obtaining numerical solutions are given in the last part of this paper.

It is well known that the Wentzel-Kramer-Brillouin (WKB) approximation, so widely used for computing the propagation constants of multimode optical waveguides, fails for the lowest order modes [4]. Analytical solutions exist for the case of a step-index guide [5], and approximate solutions for the case of parabolic index variation [6]. A number of numerical methods for field computations in the case of an arbitrary index profile have been proposed and demonstrated [7]. For further references see [3] and [7].

In Sections II and III, a formulation of the field equations as a first-order vector differential equation is introduced. The components of the four-dimensional vector are chosen to achieve simplicity of the system matrix. Its symmetry properties lead to orthogonality relations between the four independent vector solutions. Physically, they are shown to express power conservation in the radial direction. A practical consequence of these relations is a simplification of the dispersion relation of the guided modes, shown to take the form of a 2×2 determinant with real elements.

In Section IV, expressions are derived for the power flow and the group delay of a guided mode. The delay is expressed in terms of the propagation constant and the field distribution. The discussion so far is carried out without introduction of approximations.

Methods for the numerical solution of the differential equation are discussed in Section V. A power series solution of the equation is developed and the special properties of the system matrix are utilized to simplify the recursion formula for the vector coefficients of the series.

A second method, the modified staircase approximation, is based on approximating the system matrix by a constant matrix multiplied by a scalar function. The solution obtained consists of a product of matrix exponentials which are easily reduced analytically to polynomials of degree 1. This is possible because of the properties of the system matrix. The solutions have similarities with the ones obtained from the WKB method and may be regarded as generalizations of the latter. This method has been tested in a computer program and the main experience with the program is described in Section VI. The matrix formalism developed in Sections II and III has been used in another paper to solve a problem of wave propagation in fibers of crystalline materials [11].

II. Reformulation of the Field Equations

The electromagnetic field equations for monochromatic waves of frequency $\omega/2\pi$ in a source-free region may be written

$$\nabla\times\boldsymbol{E}+j\omega\mu_0\boldsymbol{H}=0 \tag{1a}$$

$$\nabla\times\boldsymbol{H}-j\omega\epsilon_0 n^2\boldsymbol{E}=0. \tag{1b}$$

We shall assume the medium to be isotropic and inhomogeneous with cylindrical symmetry. The cylindrical coordinates (r,θ,z) that are used are shown in Fig. 1. The index of refraction n is assumed to be a function of r and of ω. We attempt to find solutions of the form

$$\boldsymbol{F}(r,\theta,z)=\boldsymbol{F}(r)e^{j(\nu\theta-\beta z)}$$

where

$$\nu=0,\pm1,\pm2,\cdots \tag{2}$$

is the azimuthal index and β the constant of propagation. We shall, in the following, assume $\nu\geqslant0$ except when the contrary is explicitly stated. Equation (1) is then seen to be equivalent to four scalar differential equations of first-order and two algebraic equations. The latter may be written

$$H_r=-(1/r\omega\mu_0)(\beta rE_\theta+\nu E_z) \tag{3}$$

and

$$E_r=(1/r\omega\epsilon_0 n^2)(\beta rH_\theta+\nu H_z). \tag{4}$$

It is convenient to introduce the normalized radius and the normalized constant of propagation, given respectively as

$$s=k_0 r \quad\text{and}\quad b=\beta/k_0 \tag{5}$$

with

$$k_0=\frac{\omega}{c} \quad\text{and}\quad Z_0=(\mu_0/\epsilon_0)^{1/2}. \tag{6}$$

We further introduce the vector variable

$$\boldsymbol{w}(s)=\operatorname{col}\left(sE_\theta Z_0^{-1/2},\,E_zZ_0^{-1/2},\,sH_\theta Z_0^{1/2},\,H_zZ_0^{1/2}\right). \tag{7}$$

The four scalar differential equations contained in (1) may then be written

$$\frac{d}{ds}\boldsymbol{w}=\boldsymbol{M}\boldsymbol{w}. \tag{8}$$

Here $\boldsymbol{M}$ and $\boldsymbol{L}$ are respectively the 4×4 and 2×2 matrices given by

$$\boldsymbol{M}=\begin{pmatrix}\boldsymbol{0} & j\dfrac{1}{n^2}\boldsymbol{L}\\ -j\boldsymbol{L} & 0\end{pmatrix}$$

$$\boldsymbol{L}=\frac{1}{s}\begin{pmatrix}\nu b & \nu^2-n^2s^2\\ n^2-b^2 & -\nu b\end{pmatrix}. \tag{9}$$

The four-dimensional vector equation (8) is, together with (3) and (4), equivalent to Maxwell's equations. The vector $\boldsymbol{w}$ is determined by the field components that are tangential

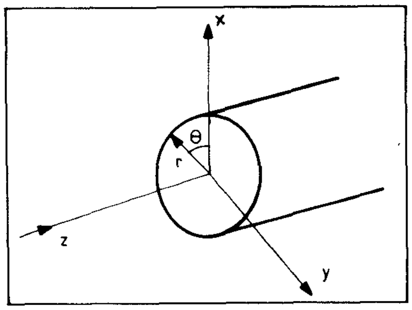

Fig. 1. Cylindrical coordinate system.

to a cylinder of radius s. As explained in [3], this choice allows continuity of $w(s)$ even in points where the refractive index $n(s)$ is discontinuous. Otherwise, the components of w have been chosen to achieve simplicity of the system matrix $\boldsymbol{M}$. The constant b will be assumed real and $\boldsymbol{M}$ is, therefore, pure imaginary.

Equation (8) always has four linearly independent solutions. It is frequently convenient to regard these as the column vectors of a 4×4 matrix $\boldsymbol{W}(s)$ which then satisfies

$$(d/ds)\boldsymbol{W}=\boldsymbol{M}\boldsymbol{W}. \tag{10}$$

Any solution of (8) may then be written as

$$w(s)=\boldsymbol{W}(s)\boldsymbol{c} \tag{11}$$

where $\boldsymbol{c}$ is a constant vector. Since the column vectors of $\boldsymbol{W}$ are linearly independent, the matrix is nonsingular and may be normalized by choice of a multiplicative constant of integration, so that it equals the unit matrix at an arbitrary point s_0. This form of the solution is denoted by $\boldsymbol{W}(s,s_0)$ and is called the matricant, the fundamental matrix or the *transition matrix* of the system. Its most important general properties are expressed by the following three relations ([8], p. 175):

$$\boldsymbol{W}(s_0,s_0)=\mathbf{1} \tag{12}$$

$$\boldsymbol{W}^{-1}(s,s_0)=\boldsymbol{W}(s_0,s) \tag{13}$$

and

$$\boldsymbol{W}(s,s')\boldsymbol{W}(s',s_0)=\boldsymbol{W}(s,s_0) \tag{14}$$

where s' is arbitrary.

We shall in the following section derive some analytical relations satisfied by the solutions of (8).

III. General Properties of the Solutions

A. The System Matrix

We first note that

$$\boldsymbol{M}^2=-\kappa^2\mathbf{1} \tag{15}$$

where

$$\kappa=\left(n^2-b^2-\nu^2/s^2\right)^{1/2}. \tag{16}$$

Aside from a factor k_0, this is the radial wavenumber component which is taken to be either positive real or positive imaginary. Thus, the matrix $\boldsymbol{M}$ is permanently degenerate, having the two double eigenvalues $+j\kappa$ and $-j\kappa$.

For any value of s for which $\kappa\neq 0$, $\boldsymbol{M}$ has a complete set of eigenvectors. The two eigenvectors corresponding to eigenvalue $+j\kappa$ span a two-dimensional vector space $\mathcal{V}_+(s)$.

The projector

$$\boldsymbol{P}=\frac{1}{2j\kappa}(j\kappa\mathbf{1}+\boldsymbol{M}) \tag{17}$$

projects any four-dimensional vector into this space. Likewise the projector

$$\boldsymbol{N}=\frac{1}{2j\kappa}(j\kappa\mathbf{1}-\boldsymbol{M}) \tag{18}$$

projects into the two-dimensional space $\mathcal{V}_-(s)$ corresponding to negative eigenvalues. Since $\boldsymbol{P}^2=\boldsymbol{P}$ and $\boldsymbol{N}^2=\boldsymbol{N}$, they are indeed projectors, and in addition $\boldsymbol{NP}=\boldsymbol{PN}=0$. The fact that $\boldsymbol{P}$ and $\boldsymbol{N}$ are the eigenprojectors of $\boldsymbol{M}$ is evident from

$$\boldsymbol{MP}=j\kappa\boldsymbol{P}\quad\text{and}\quad\boldsymbol{MN}=-j\kappa\boldsymbol{N}. \tag{19}$$

The values of s for which $\kappa=0$ are called caustics or *turning points*. Here, the projectors (17) and (18) become singular. The system matrix is nonsemisimple at these points, having only two lineary independent eigenvectors corresponding to eigenvalue zero.

In order to expose the symmetry of $\boldsymbol{M}$ we introduce

$$\boldsymbol{\sigma}=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}. \tag{20}$$

Denoting the transpose of $\boldsymbol{M}$ by $\tilde{\boldsymbol{M}}$, we find

$$\boldsymbol{\sigma M\sigma}=-\tilde{\boldsymbol{M}}^* \tag{21}$$

showing that $\boldsymbol{M}$ is *σ-skew Hermitian* ([8], p. 224). We also note that $\boldsymbol{\sigma}^2=\mathbf{1}$ and shall see that $\boldsymbol{\sigma}$ acts as a metric for the vector solutions.

B. Properties of the Solutions

The symmetry properties of $\boldsymbol{M}$ lead to orthogonality relations between the vector solutions. One consequence of these is that a solution regular at $s=0$ transmits no power in the radial direction. Let $w(s)$ be a vector solution. Then

$$\frac{d}{ds}(\tilde{w}^*\boldsymbol{\sigma}w)=\tilde{w}^*\tilde{\boldsymbol{M}}^*\boldsymbol{\sigma}w+\tilde{w}^*\boldsymbol{\sigma M}w$$

and, by invoking (21), we observe that the right-hand side is zero. Hence

$$\tilde{w}^*\boldsymbol{\sigma}w=\text{const.} \tag{22}$$

In fact, by the same method, any two solutions are seen to satisfy

$$\tilde{w}_1^*\boldsymbol{\sigma}w_2=\text{const.} \tag{23}$$

The physical meaning of the conservation theorem (22) is that power flow in the radial direction is constant, i.e., independent of s. In applying these results to the transition matrix, we find, according to (12)

$$\tilde{W}^*\sigma W = \text{const matrix} = \sigma. \tag{24}$$

Thus, W is σ-unitary, i.e.,

$$W^{-1} = \sigma\tilde{W}^*\sigma. \tag{25}$$

Two vector solutions of (8) have singularities at $s=0$. These are not acceptable as solutions of physical problems in a source-free region. The other two, say $v_1(s)$ and $v_2(s)$ may be chosen so that

$$v_1(s)s^{-\nu} \to x_0 \qquad v_2(s)s^{-\nu} \to y_0 \tag{26}$$

when

$$s \to 0.$$

Here, x_0 and y_0 are constant vectors given by (59) and (60). The vector functions $v_1(s)$ and $v_2(s)$ may then be seen to satisfy

$$\tilde{v}_1^*\sigma v_1 = \tilde{v}_2^*\sigma v_2 = \tilde{v}_1^*\sigma v_2 = 0. \tag{27}$$

To see this, observe that according to (22) and (23) each of the three products above are constant, i.e., independent of s. On the other hand, they are all zero for $s=0$ according to (26). Hence, the result (27).

C. Solutions With Real or Pure Imaginary Components

1) *Any solution $w(s)$ of the differential equation* (8) *may be written as a sum of two solutions with components that are either real or pure imaginary*: To see that this is true, let the components of $w(s)$ be

$$p_i + jq_i, \qquad i=1,\cdots,4 \tag{28}$$

where p_i and q_i are real. Then

$$w = g + h \tag{29}$$

where

$$g = \text{col}(p_1, p_2, jq_3, jq_4) \qquad h = \text{col}(jq_1, jq_2, p_3, p_4). \tag{30}$$

When inserted into the differential equation (8), (29) gives

$$\frac{d}{ds}g - Mg = -\left(\frac{d}{ds}h - Mh\right). \tag{31}$$

Inserting further for the system matrix M from (8) we observe that (31) represents four scalar equations, each equating a real component on one side to an imaginary component on the other. This is possible only if all the components are zero, i.e.,

$$\frac{d}{ds}g(s) - Mg(s) = 0$$

$$\frac{d}{ds}h(s) - Mh(s) = 0. \tag{32}$$

The above shows that $g(s)$ and $h(s)$ are themselves solutions and each of them have components that are either real or imaginary as stated above.

2) *The two solutions $v_1(s)$ and $v_2(s)$ with properties at $s=0$ as defined by* (26) *have electric and magnetic components that are respectively real and imaginary*: Consider first $v_1(s)$ which, according to assumption, tend to

$$x_0 s^{\nu}$$

for small values of s. From the series expansion of solutions discussed in Section V, it is seen that all the vector coefficients are real in their two first components and imaginary in the two others. Since s is real, the result follows as stated above. The proof for $v_2(s)$ is similar.

Solutions that are regular for $s=0$ and that decrease exponentially for large values of s are called *the guided modes* for the cylinder.

3) *A guided mode may always be chosen so that the two first components are real and the remaining two imaginary*: Any solution $w(s)$, regular at $s=0$, must be a linear combination of $v_1(s)$ and $v_2(s)$

$$w(s) = \alpha_1 v_1(s) + \alpha_2 v_2(s). \tag{33}$$

The coefficients α_1 and α_2 must then be chosen so that (33) has the prescribed exponential decrease for large values of s. However, this can only be achieved for a discrete set of values of the propagation constant b. In Section V, a derivation is given of the dispersion relation that determines these values of b and it is shown that α_1 and α_2 may be chosen real. It follows that $w(s)$ has real and imaginary components as asserted above.

IV. Power and Energy: The Group Delay of a Guided Mode

The power flow and the stored energy are important on their own account, but are discussed here also because they allow determination of the group velocity and the group delay.

A. Power flow

The total time average power flow in the radial direction per unit length is given by

$$P_T = \frac{\lambda}{4}\tilde{w}^*\sigma w \tag{34}$$

which, according to (22), is a quantity independent of s. λ is the wavelength in vacuum. The density of power flow in the azimuthal direction is

$$S_\theta = -\frac{1}{4j}\tilde{w}^*\left(\sigma\frac{\partial}{\partial\nu}M\right)w. \tag{35}$$

In the same way, the axial component of Poynting's vector is seen to be

$$S_z = \frac{1}{4js}\tilde{w}^*\left(\sigma\frac{\partial}{\partial b}M\right)w \tag{36}$$

and the total power flow of the mode in the axial direction is

$$P_z = \frac{\lambda^2}{8\pi j}\int_0^\infty \tilde{w}^*\left(\sigma\frac{\partial}{\partial b}M\right)w\,ds \tag{37}$$

where $s = k_0 r$. When the differentiation in the intergrand is carried out, (37) takes the form

$$P_z = \frac{\lambda^2}{8\pi b}\int_0^\infty \tilde{w}^* A w\,ds \tag{38}$$

where A is the real, symmetric matrix

$$A=\frac{b}{s}\begin{pmatrix} 2b & \nu & 0 & 0 \\ \nu & 0 & 0 & 0 \\ 0 & 0 & \frac{2b}{n^2} & \frac{\nu}{n^2} \\ 0 & 0 & \frac{\nu}{n^2} & 0 \end{pmatrix}. \tag{39}$$

B. The Stored Energy

To find the group velocity of a guided mode when the axial power flow is known, we need to know the energy density. In terms of the field vectors, the time average energy density at an arbitrary point may be written

$$u=\frac{1}{4}\mu_0|\boldsymbol{H}|^2+\frac{1}{4}\epsilon_0\frac{\partial}{\partial\omega}(n^2\omega)|\boldsymbol{E}|^2. \tag{40}$$

When n is differentiated with respect to ω, r (not s) should be kept constant. The term proportional to $(\partial n/\partial\omega)$ then represents the energy contribution due to the dispersion of the material. In the same way as before, we substitute $\boldsymbol{E}$ and $\boldsymbol{H}$ for the state vector $\boldsymbol{w}$ and obtain, after some manipulations

$$u=\frac{1}{4jsc}\tilde{\boldsymbol{w}}^*\boldsymbol{\sigma}\left(b\frac{\partial}{\partial b}\boldsymbol{M}+\nu\frac{\partial}{\partial\nu}\boldsymbol{M}+\lambda\frac{dn}{d\lambda}\frac{\partial}{\partial n}\boldsymbol{M}-\boldsymbol{M}\right)\boldsymbol{w}. \tag{41}$$

The total time average stored energy per unit length of the cylinder then is

$$U=2\pi\int_0^\infty ur\,dr=\frac{\lambda^2}{2\pi}\int_0^\infty us\,ds \tag{42}$$

or

$$U=\frac{\lambda^2}{8\pi jc}\left[b\int_0^\infty\tilde{\boldsymbol{w}}^*\left(\boldsymbol{\sigma}\frac{\partial}{\partial b}\boldsymbol{M}\right)\boldsymbol{w}\,ds+\int_0^\infty\tilde{\boldsymbol{w}}^*\boldsymbol{\sigma}\left(\nu\frac{\partial}{\partial\nu}\boldsymbol{M}-\boldsymbol{M}\right)\boldsymbol{w}\,ds+\lambda\int_0^\infty\frac{dn}{d\lambda}\tilde{\boldsymbol{w}}^*\left(\boldsymbol{\sigma}\frac{\partial}{\partial\nu}\boldsymbol{M}\right)\boldsymbol{w}\,ds\right]. \tag{43}$$

The first term on the right in (43) may now be eliminated by means of (37). For the second and third terms, we carry out the differentiations and define

$$\boldsymbol{E}=\frac{1}{j}\boldsymbol{\sigma}\left(\nu\frac{\partial}{\partial\nu}\boldsymbol{M}-\boldsymbol{M}\right)=\frac{1}{s}\operatorname{diag}\left(n^2-b^2,\nu^2+n^2s^2,1-(b/n)^2,s^2+(\nu/n)^2\right) \tag{44}$$

$$\boldsymbol{F}=-\frac{1}{j}\boldsymbol{\sigma}\frac{\partial}{\partial n}\boldsymbol{M}=\frac{2}{sn^3}\begin{pmatrix} n^4 & 0 & 0 & 0 \\ 0 & n^4s^2 & 0 & 0 \\ 0 & 0 & b^2 & \nu b \\ 0 & 0 & \nu b & \nu^2 \end{pmatrix}. \tag{45}$$

With this, (43) is written

$$U=\frac{b}{c}P_z+\frac{\lambda^2}{8\pi c}\cdot\left(\int_0^\infty\tilde{\boldsymbol{w}}^*\boldsymbol{E}\boldsymbol{w}\,ds-\lambda\int_0^\infty\frac{\partial n}{\partial\lambda}\cdot\tilde{\boldsymbol{w}}^*\boldsymbol{F}\boldsymbol{w}\,ds\right). \tag{46}$$

Here, the quantity b/c is the inverse of the phase velocity

$$v_p=\frac{c}{b}. \tag{47}$$

The matrices $\boldsymbol{E}$ and $\boldsymbol{F}$ are real and symmetric. They are also diagonal or near-diagonal, which simplifies the computation of the two integrals.

C. The Group Delay

Let us denote the group velocity by v_G. Its inverse is the group delay

$$\tau_G=\frac{1}{v_G}. \tag{48}$$

Under very general conditions the group velocity is identical with the energy velocity. We obtain accordingly from (46)

$$\tau_G=\frac{U}{P_z}=\frac{1}{v_p}(1+\tau_w+\tau_m) \tag{49}$$

where we have

$$\tau_m=-\lambda\frac{\int_0^\infty\frac{dn}{d\lambda}\tilde{\boldsymbol{w}}^*\boldsymbol{F}\boldsymbol{w}\,ds}{\int_0^\infty\tilde{\boldsymbol{w}}^*\boldsymbol{A}\boldsymbol{w}\,ds} \tag{50}$$

and

$$\tau_w=\frac{\int_0^\infty\tilde{\boldsymbol{w}}^*\boldsymbol{E}\boldsymbol{w}\,ds}{\int_0^\infty\tilde{\boldsymbol{w}}^*\boldsymbol{A}\boldsymbol{w}\,ds} \tag{51}$$

where τ_m is the delay due to material dispersion. Since

$$\left|\lambda\frac{dn}{d\lambda}\right|\ll 1$$

τ_m is a small quantity. The delay τ_w is the delay caused by waveguide dispersion. For optical waveguides, τ_w will also be small. This is seen by considering the numerator in (51). It evidently consists of four terms. Two of these are proportional to (b^2-n^2) which for most waveguides will be a small number. Moreover, this quantity changes sign in the region of integration, leading to partial canceling. The two remaining terms are proportional to $|E_z|^2$ and $|H_z|^2$, respectively, and are small because the axial field components of the solutions are known to be small.

The form of the expression (49) where the computed quantities τ_m and τ_w appear as small corrections, allows accurate determination of the delay without excessive requirements for the accuracy of $\boldsymbol{w}$.

V. Methods of Solution

A. Solution by Power Expansion at the Origin

The system matrix $\boldsymbol{M}$ has a simple pole at $s=0$. For small values of s, $\boldsymbol{M}$ tends to

$$\boldsymbol{M} \rightarrow \frac{1}{s}\boldsymbol{R} \tag{52}$$

where $\boldsymbol{R}$ is the residue matrix

$$\boldsymbol{R} = \begin{pmatrix} \mathbf{0} & j\frac{1}{n_0^2}\boldsymbol{L}_0 \\ -j\boldsymbol{L}_0 & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{L}_0 = \begin{pmatrix} \nu b & \nu^2 \\ n_0^2 - b^2 & -\nu b \end{pmatrix} \tag{53}$$

with

$$n_0 = n(0). \tag{54}$$

The residue matrix has the two double eigenvalues $+\nu$ of $-\nu$ and it is easy to see that

$$\boldsymbol{R}^2 = \nu^2 \mathbf{1}. \tag{55}$$

For sufficiently small values of s, the differential equation (10) approaches the equation

$$\frac{d}{ds}\boldsymbol{W} = \frac{1}{s}\boldsymbol{R}\boldsymbol{W}. \tag{56}$$

Since $\boldsymbol{R}$ is a constant matrix, (56) is Euler's equation with the solution

$$\boldsymbol{W} = e^{\boldsymbol{R}\ln s}. \tag{57}$$

When $\nu \geqslant 1$, this may, through use of (55), be reduced to

$$\boldsymbol{W} = \frac{1}{2}\left(\mathbf{1} + \frac{1}{\nu}\boldsymbol{R}\right)s^{\nu} + \frac{1}{2}\left(\mathbf{1} - \frac{1}{\nu}\boldsymbol{R}\right)s^{-\nu}. \tag{58}$$

The matrix factors on the right-hand side are projectors of rank 2. It follows that there are two pairs of vector solutions, proportional to s^{ν} and $s^{-\nu}$, respectively.

The vector space corresponding to s^{ν} is spanned by the vectors

$$\boldsymbol{x}_0 = \mathrm{col}\left(b\nu/n_0^2, 1 - b^2/n_0^2, -j\nu, 0\right) \tag{59}$$

$$\boldsymbol{y}_0 = \mathrm{col}\left(-\nu/n_0^2, 0, j\nu b/n_0^2, j\left(1 - b^2/n_0^2\right)\right). \tag{60}$$

These are linearly independent eigenvectors of $\boldsymbol{R}$ corresponding to the eigenvalue $+\nu$. It is readily seen that they satisfy

$$\begin{aligned} \tilde{\boldsymbol{x}}_0^* \boldsymbol{\sigma} \boldsymbol{x}_0 &= 0 \\ \tilde{\boldsymbol{y}}_0^* \boldsymbol{\sigma} \boldsymbol{y}_0 &= 0 \\ \tilde{\boldsymbol{x}}_0^* \boldsymbol{\sigma} \boldsymbol{y}_0 &= 0. \end{aligned} \tag{61}$$

For $\nu = 0$, (57) reduces to

$$\boldsymbol{W} = (\mathbf{1} + \boldsymbol{R}\ln s). \tag{62}$$

Here, one pair of vector solutions tends to a finite value at the origin. The other pair has a logarithmic singularity at $s=0$.

The former pair is found by putting $\nu = 0$ in (59) and (60). For any value of ν, the solutions have, as one might expect, the same behavior as the Bessel functions close to $s=0$.

Since only the solutions being finite at $s=0$ can be part of the guided modes, it is computationally advantageous to work with the vector wave equation (8) rather than the matrix equation, and we shall develop the solution of the former in terms of a power series.

1) *Power series solution of the vector wave equation*: We attempt to write a solution $\boldsymbol{w}$ of (8) as

$$\boldsymbol{w}(s) = \sum_{p=0}^{\infty} \boldsymbol{w}_p s^{\nu+p}. \tag{63}$$

Here, the $\boldsymbol{w}_p$'s are vector coefficients to be determined and

$$\nu = 0, 1, 2, \cdots$$

is as before the azimuthal wavenumber. By writing $\boldsymbol{w}(s)$ in this form we have excluded the solutions tending to infinity at the origin.

We also write the system matrix

$$\boldsymbol{M} = \frac{1}{s}\boldsymbol{R} + \sum_{q=0}^{\infty} \boldsymbol{M}_q s^q. \tag{64}$$

This power expansion of the system matrix may be found by substitution of the functions $n^2(s)$ and $1/n^2(s)$ for their power series in (9).

When the power series for $\boldsymbol{w}$ and $\boldsymbol{M}$ are inserted into the differential equation (8), we obtain by equating terms of the same power

$$(p+\nu)\boldsymbol{w}_p = \sum_{q=-1}^{p-1} \boldsymbol{M}_q \boldsymbol{w}_{p-q-1}. \tag{65}$$

Here, we have put

$$\boldsymbol{R} = \boldsymbol{M}_{-1}. \tag{66}$$

For $p = 0$ we obtain from (65)

$$\nu \boldsymbol{w}_0 = \boldsymbol{R}\boldsymbol{w}_0 \tag{67}$$

which shows that $\boldsymbol{w}_0$ is an eigenvector corresponding to eigenvalue $+\nu$ of the residue matrix $\boldsymbol{R}$. This means that $\boldsymbol{w}_0$ must be some linear combination of the vectors $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ given by (59) and (60). Hence

$$\boldsymbol{w}_0 = \alpha_1 \boldsymbol{x}_0 + \alpha_2 \boldsymbol{y}_0. \tag{68}$$

For the guided modes, the ratio α_1/α_2 is determined by the boundary condition at infinity. A slight rearranging of terms in (65) gives

$$[(p+\nu)\mathbf{1} - \boldsymbol{R}]\boldsymbol{w}_p = \boldsymbol{M}_0\boldsymbol{w}_{p-1} + \boldsymbol{M}_1\boldsymbol{w}_{p-2} + \cdots + \boldsymbol{M}_{p-1}\boldsymbol{w}_0. \tag{69}$$

With the relation (55) in mind, we readily see that the matrix on the left may be inverted analytically and obtain

$$\boldsymbol{w}_p = \frac{1}{p(p+2\nu)}[(p+\nu)\mathbf{1} + \boldsymbol{R}](\boldsymbol{M}_0\boldsymbol{w}_{p-1} + \cdots + \boldsymbol{M}_{p-1}\boldsymbol{w}_0) \tag{70}$$

which allows successive computing of all the vector coefficients starting out with $\boldsymbol{w}_0$ as given by (68). No matrix inversion is needed in computing the coefficients.

2) *The case of constant index $n=n_2$:* When n is a constant, the only nonzero coefficients in the expansion (64) are $\boldsymbol{R}$ and $\boldsymbol{M}_1$. The recursion formula then reduces to

$$w_p=\frac{1}{p(p+2\nu)}[(p+\nu)\mathbf{1}+\boldsymbol{R}]\boldsymbol{M}_1 w_{p-2}. \quad (71)$$

The series (65) may now be summed analytically to give the well-known result of Schnitzer [5]. The four linearly independent solutions of (8) are

$$\boldsymbol{u}_1=\begin{pmatrix} -j\dfrac{\nu b}{n_2\rho}K_\nu \\ j\dfrac{\rho}{n_2}K_\nu \\ -n_2 sK_\nu' \\ 0 \end{pmatrix} \qquad \boldsymbol{u}_2=\begin{pmatrix} jsK_\nu' \\ 0 \\ \dfrac{\nu b}{\rho}K_\nu \\ -\rho K_\nu \end{pmatrix} \quad (72)$$

$$\boldsymbol{u}_3=\begin{pmatrix} \dfrac{\nu b}{n_2\rho}I_\nu \\ -\dfrac{\rho}{n_2}I_\nu \\ -jn_2 sI_\nu' \\ 0 \end{pmatrix} \qquad \boldsymbol{u}_4=\begin{pmatrix} -sI_\nu' \\ 0 \\ j\dfrac{\nu b}{\rho}I_\nu \\ -j\rho I_\nu \end{pmatrix}. \quad (73)$$

The argument of the modified Bessel functions is ρs

$$K_\nu=K_\nu(\rho s) \qquad I_\nu=I_\nu(\rho s).$$

In the above, we have put

$$\left(b^2-n_2^2\right)^{1/2}=\rho \quad (74)$$

assuming $n_2<b$. The set of vectors $\boldsymbol{u}_m$ form a complete set of solutions and are related by

$$\tilde{\boldsymbol{u}}_1{}^*\boldsymbol{\sigma}\boldsymbol{u}_3=\tilde{\boldsymbol{u}}_2^*\boldsymbol{\sigma}\boldsymbol{u}_4=1. \quad (75a)$$

For the other products, we have

$$\tilde{\boldsymbol{u}}_p^*\boldsymbol{\sigma}\boldsymbol{u}_q=0. \quad (75b)$$

The correctness of (75a) follows from a well-known result for the Wronskian of the modified Bessel functions ([9], p. 375). These orthogonal relations will be seen to be important in the following for obtaining a simple dispersion relation for a waveguide with arbitrary radial index variation.

B. Solution by a Modified Staircase Approximation

The index is assumed to be an arbitrary function of s in the region

$$0\leqslant s\leqslant a \quad (76)$$

and a constant, equal to n_2, outside this region. It follows that for

$$s\geqslant a \quad (77)$$

the solutions (72) and (73) apply. For very small values of s, the solution is determined by the first term of the series (63), but the series will in general converge too slowly to be of much practical use for computing the solution for larger values of s.

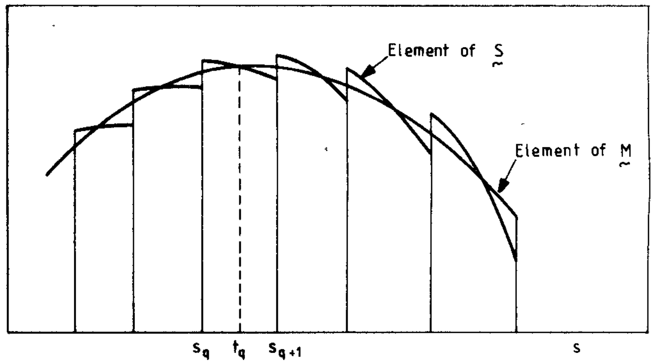

Fig. 2. Approximate discontinuous system functions.

Vigants and Schlesinger [3] make use of a staircase approximation in which the continuous function $n(s)$ is replaced by a piecewise constant function. The efficiency of the method has been studied in greater detail by Yeh and Lindgren [2]. A weak point in the method is that use of the modified Bessel function

$$K_\nu\left(s\sqrt{b^2-n^2}\right)$$

in expressing the approximate solution introduces a singularity at the value of s for which

$$n^2(s)=b^2. \quad (78)$$

The exact solution has no such singularity.

In the following, we shall sketch a modified staircase approximation in which use is made of the fact that the special properties of $\boldsymbol{M}$ make the computation of the matrix exponential particularly easy.

Let t_q be the midpoint of the interval s_q, s_{q+1}. Within this interval we approximate the system matrix $\boldsymbol{M}(s)$ by

$$\boldsymbol{S}(s)=\frac{\kappa(s)}{\kappa(t_q)}\boldsymbol{M}(t_q). \quad (79)$$

It is seen that the approximate system matrix $\boldsymbol{S}$ in each interval is equal to a constant matrix times a scalar function. This scalar function is chosen so that $\boldsymbol{S}(s)$ has the correct eigenvalues everywhere in the interval, whereas the eigenvectors (or eigenprojectors) are correct in the midpoint of each interval only (Fig. 2).

Introducing

$$\boldsymbol{D}(s)=\boldsymbol{M}(s)-\boldsymbol{S}(s) \quad (80)$$

we may write the differential equation (10) as

$$\frac{d}{ds}\boldsymbol{W}-\boldsymbol{S}(s)\boldsymbol{W}=\boldsymbol{D}(s)\boldsymbol{W}. \quad (81)$$

The term on the right-hand side is zero at the midpoint t_q of each interval. Its matrix norm is then small over the entire interval, provided that the latter is chosen to be sufficiently narrow.

The homogeneous equation corresponding to (81) is

$$\frac{d}{ds}\boldsymbol{W}_0-\boldsymbol{S}(s)\boldsymbol{W}_0=0. \quad (82)$$

The resulting transition matrix $\boldsymbol{W}_0$, evidently a first approximation to $\boldsymbol{W}$, is seen to be

$$\boldsymbol{W}_0(s,\xi)=\exp\left[\boldsymbol{M}(t_q)\int_\xi^s(\kappa(s)/\kappa(t_q))\,ds\right] \tag{83}$$

where s and ξ belong to the interval.

When use is made of the projectors defined in (17) and (18)

$$\begin{aligned}\boldsymbol{P}&=\boldsymbol{P}(t_q)\\ \boldsymbol{N}&=\boldsymbol{N}(t_q)\end{aligned} \tag{84}$$

we may write

$$\boldsymbol{M}(t_q)=j\kappa(t_q)(\boldsymbol{P}-\boldsymbol{N}). \tag{85}$$

We make use of the relations

$$(\boldsymbol{P}-\boldsymbol{N})^p=\boldsymbol{P}+(-1)^p\boldsymbol{N} \tag{86}$$

and

$$\mathbf{1}=\boldsymbol{P}+\boldsymbol{N} \tag{87}$$

and obtain from the Taylor expansion of (83)

$$\boldsymbol{W}_0(s,\xi)=\boldsymbol{P}(t_q)\exp\left(j\int_\xi^s\kappa\,ds\right)+\boldsymbol{N}(t_q)\exp\left(-j\int_\xi^s\kappa\,ds\right). \tag{88}$$

This expression is evidently quite similar to the one that results from applying the WKB approximation to the scalar wave equation [4]. An important difference is that here the polarizations of the two waves appear directly in the expression.

Let us introduce

$$\phi(s,\xi)=\int_\xi^s|\kappa(s)|\,ds. \tag{89}$$

For the region in which $\kappa(s)$ is real, (88) may then be written

$$\boldsymbol{W}_0(s,\xi)=\mathbf{1}\cos\phi(s,\xi)+\frac{1}{|\kappa(t_q)|}\boldsymbol{M}(t_q)\sin\phi(s,\xi). \tag{90}$$

For the region in which κ is imaginary, we find similarly

$$\boldsymbol{W}_0(s,\xi)=\mathbf{1}\cosh\phi(s,\xi)+\frac{1}{|\kappa(t_q)|}\boldsymbol{M}(t_q)\sinh\phi(s,\xi). \tag{91}$$

Putting now

$$s=s_{q+1}\text{ and }\xi=s_q$$

the first approximation of the transition matrix from s_q to s_{q+1} is

$$\boldsymbol{W}_0(s_{q+1},s_q)$$

as given by (90) and (91) for real and imaginary κ's, respectively.

1) Improvement of the Accuracy: Returning now to the exact equation (81), we may improve the accuracy by making use of a perturbation type of solution. Using a well-known result ([8], p. 187) we obtain

$$\boldsymbol{W}(s,\xi)=\boldsymbol{W}_0(s,\xi)+\int_\xi^s\boldsymbol{W}_0(s,s')\boldsymbol{D}(s')\boldsymbol{W}(s',\xi)\,ds'. \tag{92}$$

This is an integral equation for $\boldsymbol{W}(s,\xi)$. Since $\boldsymbol{D}(s')$ is small everywhere in the interval and zero at the midpoint, a correction term is obtained by replacing $\boldsymbol{W}$ under the integral sign by $\boldsymbol{W}_0$. A three point Simpson method for evaluation of the integral then gives

$$\boldsymbol{W}\approx\boldsymbol{W}_0+\frac{s_{q+1}-s_q}{6}\left[\boldsymbol{W}_0\boldsymbol{D}(s_q)+\boldsymbol{D}(s_{q+1})\boldsymbol{W}_0\right] \tag{93}$$

where we have put

$$\begin{aligned}\boldsymbol{W}_0(s_{q+1},s_q)&=\boldsymbol{W}_0\\ \boldsymbol{W}(s_{q+1},s_q)&=\boldsymbol{W}.\end{aligned} \tag{94}$$

The transition matrix over a distance of several intervals is now found by means of the product rule shown in (14).

C. Dispersion Relation for the Guided Modes

Assume that we by some approximate method have found two vector solutions $\boldsymbol{v}_1(s)$ and $\boldsymbol{v}_2(s)$ for the region $0\leqslant s\leqslant a$. The two functions are well behaved at $s=0$. In the region $s\geqslant a$, both solutions may, as we have seen, be expressed in terms of the vector functions (72) and (73). We require that some linear combination of $\boldsymbol{v}_1(s)$ and $\boldsymbol{v}_2(s)$

$$\boldsymbol{w}(s)=\alpha_1\boldsymbol{v}_1(s)+\alpha_2\boldsymbol{v}_2(s) \tag{95}$$

tend to zero at $s\to\infty$. This means that $\boldsymbol{w}(a)$ has zero projection into the vector space spanned by $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ in (73). When use is made of the orthogonality relations (75), this condition leads to

$$\begin{aligned}\alpha_1\tilde{\boldsymbol{u}}_1^*(a)\boldsymbol{\sigma}\boldsymbol{v}_1(a)+\alpha_2\tilde{\boldsymbol{u}}_1^*(a)\boldsymbol{\sigma}\boldsymbol{v}_2(a)&=0\\ \alpha_1\tilde{\boldsymbol{u}}_2^*(a)\boldsymbol{\sigma}\boldsymbol{v}_1(a)+\alpha_2\tilde{\boldsymbol{u}}_2^*(a)\boldsymbol{\sigma}\boldsymbol{v}_2(a)&=0\end{aligned} \tag{96}$$

or

$$\begin{vmatrix}\tilde{\boldsymbol{u}}_1^*(a)\boldsymbol{\sigma}\boldsymbol{v}_1(a), & \tilde{\boldsymbol{u}}_1^*(a)\boldsymbol{\sigma}\boldsymbol{v}_2(a)\\ \tilde{\boldsymbol{u}}_2^*(a)\boldsymbol{\sigma}\boldsymbol{v}_1(a), & \tilde{\boldsymbol{u}}_2^*(a)\boldsymbol{\sigma}\boldsymbol{v}_2(a)\end{vmatrix}=0. \tag{97}$$

This is the dispersion relation which determines the normalized constant of propagation b. Although different in form, the relation is equivalent to the characteristic equation derived by Vigant and Schlesinger [3]. For the special case of $n(s)=\text{const}$, (97) reduces to the well-known dispersion relation for a step-index guide ([10], p. 296). The determinant (97) has real terms only. This follows from the fact that in $\boldsymbol{v}_1(a)$ and $\boldsymbol{v}_2(a)$ the electric and magnetic components are respectively real and imaginary. The constant b must be found through numerical search for the value that satisfies (97).

VI. Experience with the Computer Program

A computer program was made based on the modified staircase method, described in Section V-B. The program computes propagation constant and field distribution for guided modes of a cylinder with arbitrary radial index variation. First two vector solutions are computed starting respectively with

$$\boldsymbol{x}_0 s^\nu \text{ and } \boldsymbol{y}_0 s^\nu$$

for a small value of s, e.g., $s = 10^{-6}$. Here, $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ are the vectors given by (59) and (60). Then successive values of the solutions are computed in steps by means of the expression (94). It was found advantageous to use the correction term (93) because this allowed the use of larger steps without serious reduction in accuracy. The program was tested by computing fields and propagation constant for the HE_{11} mode of a step-index fiber. Since the solutions for this case are well known, the accuracy could be tested. With steps of s equal to 1, i.e., about 6 steps per wavelength, the relative error in the fields per step was smaller than 10^{-6} over most of the region. As a further check, the cutoff value of (ρs) for the H_{01} mode was computed. The correct value is known to be the first zero of $J_0(z)$. The error in the computed result was $1.2 \cdot 10^{-5}$.

If the expression for the transition matrix is expanded in terms of the length of the interval $(s_{q+1} - s_q)$, the first errors occur in the fourth-order term. Also, $\boldsymbol{W}$, as given by (93), is σ-unitary up to terms of order 4, i.e.,

$$\boldsymbol{\sigma}\tilde{\boldsymbol{W}}^*\boldsymbol{\sigma}\boldsymbol{W} - \mathbf{1} = \text{const}\,(s_{q+1} - s_q)^4.$$

This makes the numerical method developed here to some extent self-controlling. Whenever the correction term destroys the unitary property of the transition matrix, the errors are significant. The easiest way of obtaining this control is to check that the two vector solutions satisfy the orthogonality relations (27).

The program was developed in the APL language on an IBM 5100 desk top computer. The storage capacity needed by the total program was approximately 16 kbytes.

Acknowledgment

Part of the work reported in this paper was done while the author was Visiting Professor at the University of California, Santa Barbara. The support and hospitality of the UCSB is gratefully acknowledged.

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